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The $W(sl(N+3), sl(3))$ algebras and their contractions to W_3 *

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Abstract

We construct the nonlinear $W(sl(N+3), sl(3))$ algebras and find the spectrum of values of the central charge that gives rise, by contracting the $W(sl(N+3), sl(3))$ algebras, to a W_3 algebra belonging to the coset $W((sl(N+3), sl(3))/(u(1) \oplus sl(N))$. Part of the spectrum was conjectured before, but part of it is given here for the first time. Using the tool of embedding the $W(sl(N+3), sl(3))$ algebras into linearizing algebras, we construct new realizations of W_3 modulo null fields. The possibility to predict, within the conformal linearization framework, the central charge spectrum for minimal models of the nonlinear $W(sl(N+3), sl(3))$ algebras is discussed at the end.

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*Dedicated to the memory of Victor I. Ogievetsky, as the expression of our respectful admiration and deep sorrow.

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1 Introduction

W-algebras were introduced in 1985 by Zamolodchikov [1], in terms of conformal models, exhibiting new symmetries generated by currents with conformal spin higher than 2. Owing to the nonlinearity of W-algebras, the important task of constructing their realizations in terms of free fields or affine currents cannot be carried out straightforwardly.

One of the possible ways to construct the realizations of W algebras is the conformal linearization procedure [2, 3, 4, 5, 6]. The main idea of this approach is to embed the nonlinear W algebra as a subalgebra in some linear conformal algebra W^{lin}. Once this is done, then each realization of the linear algebra W^{lin} gives rise to a realization of W.

In [6] we constructed explicitly the nonlinear algebras $W(sl(4), sl(3))$, $W(sl(3|1), sl(3))$ and obtained their realizations in terms of currents spanning the corresponding linearizing conformal algebras. The specific structure of these algebras allowed us to construct realizations - modulo null fields - of the W_3 algebra that lies in the cosets $W(sl(4), sl(3))/u(1)$ and $W(sl(3|1), sl(3))/u(1)$. In such null-fields realizations the OPE of two spin-3 currents contains a spin-4 operator which is *null*, in the sense that there is no central term in the OPE of two such spin-4 currents. This occurs only for a discrete spectrum of $W(sl(4), sl(3))$ - respectively $W(sl(3|1), sl(3))$ - central charges, which allows us to consider a vanishing value for such spin-4 operators and reduces the $W(sl(4), sl(3))$ - respectively $W(sl(3|1), sl(3))$ - algebra to its W_3 contraction. For this spectrum the realizations of $W(sl(4), sl(3))$ - respectively $W(sl(3|1), sl(3))$ - algebras provide null-fields realizations of W_3.

In this Letter we follow a similar procedure, in order to obtain new null-fields realization of W_3 from linearizing $W((sl(N+3), sl(3))$ algebras. In particular, we determine the central charge spectrum for the nonlinear $W(sl(N+3), sl(3))$ algebras, when these algebras are contracted to the W_3 algebra lying in the coset $W((sl(N+3), sl(3))/(u(1) \oplus sl(N))$. The completion of our task, namely the explicit construction of free-fields realizations - modulo null fields - for the W_3 contraction of $W(sl(N+3), sl(3))$ algebras, becomes straightforward when the conformal linearization procedure is applied to the latter algebras. Indeed, it turns out that the nonlinear $W(sl(N+3), sl(3))$ algebras can be embedded into some linear conformal algebras with a finite number of generators related to the currents of the nonlinear algebra by an *invertible* transformation. Hence it follows that any realization of the linearizing $W(sl(N+3), sl(3))$ algebras yields a realization of the nonlinear $W(sl(N+3), sl(3))$ ones.

The outline of the Letter is as follows.

We start in Section 2 with presenting the nonlinear $W(sl(N+3), sl(3))$ algebras. Their explicit structure appears here for the first time. The importance of the algebras constructed in this Section can be appreciated if one recalls that further examples of analogous algebras whose OPEs explicitly depend, in addition to the level K of the algebra, on one more parameter (in our case denoted by N) are very few. They include the Knizhnik-Bershadsky algebras, as well as the quasi superconformal algebras. In Section 3, after recalling briefly the method used to construct null-fields realizations of W_3, we consider the exceptional values of the central charges of the nonlinear $W(sl(N+3), sl(3))$ algebras for which the composite spin-4 field appearing in the spin-3 - spin-3 OPE becomes a null field. Also we discuss the properties of the spectrum and point out the result that one central charge value corresponds to three distinct representations of the contracted algebra W_3. We then proceed in Section 4 to construct the corresponding linear conformal algebras with a finite number of currents. The resulting expressions giving, in terms

these currents, those spanning the nonlinear algebras immediately provide the realizations of $W(sl(N+3), sl(3))$. The constructed expressions for the currents of the nonlinear algebras in terms of the linearizing algebras currents yield, for the specific values of the central charge, the realization of W_3 algebra modulo null fields.

We will finish this Letter with some concluding remarks and a short discussion of further developments, including applications to the central charge spectrum for minimal models of the nonlinear $W(sl(N+3), sl(3))$ algebras.

2 Nonlinear $W(sl(N+3), sl(3))$ algebras

In this Section we give explicitly for the first time the structure of the nonlinear $W(sl(N+3), sl(3))$ algebras in the quantum case. This result provides a new example of algebras that depend on two parameters, i.e. the level K and N, and possess the $SL(N)$ automorphism. The constructed $W(sl(N+3), sl(3))$ algebras are analogous in this sense to the case of the Knizhnik-Bershadsky and quasi superconformal algebras.

Let us first of all describe the conformal spin content of these algebras [7]. They are spanned, in addition to the spin-2 stress-tensor T, by the following currents: the $sl(N)$ and $u(1)$ affine spin-1 currents J_a^b and U, the commuting-with-them spin-3 current W and two multiplets of spin-2 currents $G_a, \overline{G}^a$ with opposite $u(1)$ charges, which belong to the fundamental and its conjugated representations of $sl(N)$. In the basis where these bosonic currents $(U, J_a^b, G_a, \overline{G}^a, W)$ are all primary fields with respect to T, the singular OPEs of the nonlinear $W(sl(N+3), sl(3))$ algebras read

$$
\begin{aligned}
T(z_1)T(z_2) &= \frac{c_t}{2z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \quad T(z_1)J_a^b(z_2) = \frac{J_a^b}{z_{12}^2} + \frac{J_a^{b\prime}}{z_{12}} \quad , \\
T(z_1)U(z_2) &= \frac{U}{z_{12}^2} + \frac{U'}{z_{12}} \quad , \quad T(z_1)W(z_2) = \frac{3W}{z_{12}^2} + \frac{W'}{z_{12}} \quad , \\
T(z_1)G_a(z_2) &= \frac{2G_a}{z_{12}^2} + \frac{G_a'}{z_{12}} \quad , \quad T(z_1)\overline{G}^a(z_2) = \frac{2\overline{G}^a}{z_{12}^2} + \frac{\overline{G}^{a}{}'}{z_{12}} \quad , \\
U(z_1)U(z_2) &= \frac{c_u}{z_{12}^2} \quad , \quad U(z_1)G_a(z_2) = -\frac{2G_a}{z_{12}} \quad , \quad U(z_1)\overline{G}^a(z_2) = \frac{2\overline{G}^a}{z_{12}} \quad , \\
J_a^b(z_1)J_c^d(z_2) &= K(\delta_a^d\delta_c^b - \frac{1}{N}\delta_a^b\delta_c^d)\frac{1}{z_{12}^2} + (\delta_c^b J_a^d - \delta_a^d J_c^b)\frac{1}{z_{12}} \quad , \\
J_a^b(z_1)G_c(z_2) &= (\delta_c^b G_a - \frac{1}{N}\delta_a^b G_c)\frac{1}{z_{12}} \quad , \quad J_a^b(z_1)\overline{G}^c(z_2) = (-\delta_a^c\overline{G}^b + \frac{1}{N}\delta_a^b\overline{G}^c)\frac{1}{z_{12}} \quad , \\
G_a(z_1)\overline{G}^b(z_2) &= \delta_a^b\frac{c_g}{z_{12}^4} + (\alpha_1\delta_a^b U + \alpha_2 J_a^b)\frac{1}{z_{12}^3} + \Big[\frac{1}{3}\delta_a^b T + \alpha_3\delta_a^b(U\ U) + \frac{\alpha_1}{2}\delta_a^b U' + \alpha_4(J_a^c\ J_c^b) \\
&+ \frac{\alpha_1}{K}(U\ J_a^b) + \alpha_5\delta_a^b(J_c^d\ J_d^c) + K\alpha_4 J_a^{b\prime}\Big]\frac{1}{z_{12}^2} + \Big[\delta_a^b W + \frac{1}{6}\delta_a^b T' + \alpha_6\delta_a^b(T\ U) \\
&+ \alpha_7\delta_a^b(U\ (U\ U)) + \alpha_3\delta_a^b(U'\ U) + \alpha_8\delta_a^b U'' + \alpha_9\delta_a^b(U\ (J_c^d\ J_d^c)) \\
&+ \alpha_{10}\delta_a^b(J_c^d\ (J_d^e\ J_e^c)) + \alpha_{11}\delta_a^b(J_c^{d\prime}\ J_d^c) + \frac{1}{3K}(T\ J_a^b) + \frac{\alpha_3}{K}(U\ (U\ J_a^b)) \\
&+ \alpha_{12}(U\ (J_a^c\ J_c^b)) + \alpha_{13}(J_a^b\ (J_c^d\ J_d^c)) + 2\alpha_{13}(J_a^c\ (J_c^d\ J_d^b)) + \frac{\alpha_1}{2K}(U'\ J_a^b)
\end{aligned}
$$

$$+K\alpha_{12}(U\ J_a^{b\prime}) + \alpha_{14}(J_a^{c\prime}\ J_c^b) + 2K\alpha_{13}(J_a^c\ J_c^{b\prime}) + \alpha_{15}J_a^{b\prime\prime}\Big]\frac{1}{z_{12}} \quad ,$$

$$\begin{aligned}
G_a(z_1)W(z_2) &= \frac{\beta_1 G_a}{z_{12}^3} + \Big[\beta_2 G_a{}' + \beta_3(U\ G_a) + \beta_4(J_a^b\ G_b)\Big]\frac{1}{z_{12}^2} \\
&+ \Big[\beta_5(T\ G_a) + \beta_6(U\ (U\ G_a)) + \beta_7(U\ (J_a^b\ G_b)) + \beta_8(U'\ G_a) + \beta_9(U\ G_a{}') \\
&+\beta_{10}(J_a^b\ (J_b^c\ G_c)) + \beta_{11}(J_b^c\ (J_c^b\ G_a)) + \beta_{12}(J_a^{b\prime}\ G_b) + \beta_{13}(J_a^b\ G_b{}') \\
&+\beta_{14}G_a{}''\Big]\frac{1}{z_{12}} \quad , \\
\overline{G}^a(z_1)W(z_2) &= \frac{\beta_1\overline{G}^a}{z_{12}^3} + \Big[\beta_2\overline{G}^{a\prime} - \beta_3(U\ \overline{G}^a) - \beta_4(J_b^a\ \overline{G}^b)\Big]\frac{1}{z_{12}^2} \\
&+ \Big[\beta_5(T\ \overline{G}^a) + \beta_6(U\ (U\ \overline{G}^a)) + \beta_7(U\ (J_b^a\ \overline{G}^b)) - \beta_8(U'\ \overline{G}^{a\prime}) - \beta_9(U\ \overline{G}^{a\prime}) \\
&+\beta_{10}(J_b^c\ (J_c^a\ \overline{G}^b)) + \beta_{11}(J_b^c\ (J_c^b\ \overline{G}^a)) - \beta_{12}(J_a^{b\prime}\ \overline{G}^b)) - \beta_{13}(J_a^b\ \overline{G}^{b\prime}) \\
&+\beta_{14}\overline{G}^{a\prime\prime}\Big]\frac{1}{z_{12}} \quad , \\
W(z_1)W(z_2) &= \epsilon\Bigg\{\frac{c_w}{6z_{12}^6} + T_w\frac{1}{z_{12}^4} + \frac{1}{2}T_w'\frac{1}{z_{12}^3} \\
&+ \left[U_4 + \frac{16}{22+5c_w}\left((T_w\ T_w) - \frac{3}{10}T_w''\right) + \frac{3}{20}T_w''\right]\frac{1}{z_{12}^2} \\
&+ \frac{1}{2}\left[U_4 + \frac{16}{22+5c_w}\left((T_w\ T_w) - \frac{3}{10}T_w''\right) + \frac{1}{15}T_w''\right]'\frac{1}{z_{12}}\Bigg\},
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
T_w &= T - \frac{(U\ U)}{2c_u} - \frac{(J_a^b\ J_b^a)}{2(K+N)} \quad , \\
c_w &= -\frac{(2K+N)(-1+3K+2N)(1+4K+3N)}{(K+N)(K+N+1)} \quad , \\
U_4 &= \epsilon_1 W' + \epsilon_2(W\ U) + \epsilon_3(G_a\ \overline{G}^a) + \epsilon_4(T\ T) + \epsilon_5(T\ (U\ U)) + \epsilon_6(T\ (J_a^b\ J_b^a)) \\
&+\epsilon_7(T'\ U) + \epsilon_8(T\ U') + \epsilon_9 T'' + \epsilon_{10}(U\ (U\ (U\ U))) + \epsilon_{11}(U'\ (U\ U)) \\
&+\epsilon_{12}(U\ (U\ (J_a^b\ J_b^a))) + \epsilon_{13}(U''\ U) + \epsilon_{14}(U'\ U') + \epsilon_{15}U''' + \epsilon_{16}(U\ (J_a^b\ (J_b^c\ J_c^a))) \\
&+\epsilon_{17}(U'\ (J_a^b\ J_b^a)) + \epsilon_{18}(U\ (J_a^{b\prime}\ J_b^a)) + \epsilon_{19}(J_a^b\ (J_b^c\ (J_c^d\ J_d^a))) + \epsilon_{20}(J_a^b\ (J_b^a\ (J_c^d\ J_d^c))) \\
&+\epsilon_{21}(J_a^{b\prime}\ (J_b^c\ J_c^a)) + \epsilon_{22}(J_a^{b\prime}\ J_b^{a\prime}) + \epsilon_{23}(J_a^{b\prime\prime}\ J_b^a) \\
&-\frac{16}{22+5c_w}\left[(T_w\ T_w) - \frac{3}{10}T_w''\right] - \frac{3}{20}T_w'' \quad .
\end{aligned} \tag{2.2}$$

Here the indices $a, b, ...$ run over the following ranges $1 \le a, b \le N$, and the $c_{t,u,g}$, α, β and ϵ coefficents are found in Table 1 (see appendix) in terms of the level K of $sl(N)$ algebra. Although the coefficient ϵ could be set to unity provided the current $W(z)$ is rescaled, we keep it for convenience, since this makes the coefficients simpler.

Inspecting the expressions given in Table 1 makes it evident that the OPE $T(z_1)T(z_2)$ (2.1) is singular for $K = -1-N$. In fact, the $W(sl(N+3), sl(3))$ algebras are not defined for this level value[1] and we take $K \neq -1-N$. In order to remove from the OPEs (2.1) the additional

[1]After rescaling their currents, the algebras do not contain the Virasoro subalgebra.

singularities occurring in some of the coefficients of Table 1 for the level values $K=-r$, with $r=0,N,2N/3$, we must redefine the generators of the $W(sl(N+3),sl(3))$ algebras (2.1) as follows:

$$W = \frac{1}{K+r}\widetilde{W} \quad , \quad G_a = \frac{1}{\sqrt{K+r}}\tilde{G}_a \quad , \quad \overline{G}^a = \frac{1}{\sqrt{K+r}}\tilde{\overline{G}}^a \quad , \quad r=0,N \quad , \tag{2.3}$$

$$W = \frac{1}{(K+r)^2}\widetilde{W} \quad , \quad G_a = \frac{1}{K+r}\tilde{G}_a \quad , \quad \overline{G}^a = \frac{1}{K+r}\tilde{\overline{G}}^a \quad , \quad r=\frac{2N}{3} \quad . \tag{2.4}$$

After the above redefinitions the OPEs (2.1) describe, for $K=-r$, some uninteresting algebras obtained as a contraction of $W(sl(N+3),sl(3))$. For the first two levels $K=0,-N$ the central charge c_u does not vanish, whereas $c_u=0$ at $K=-2N/3$. In all three $K=-r$ cases one has $c_g=\epsilon c_w=0$ and $c_t\neq 0$. Hence the algebra described by the OPEs (2.1) exists for all values of the level parameter K, except $K=-1-N$. However one has to keep in mind that the algebras of interest to us correspond to $K+r\neq 0$.

Let us note that the spin-4 current $U_4(z)$ is defined to be primary with respect to the stress tensor $T_w(z)$[2]

$$T_w(z_1)T_w(z_2) = \frac{c_w}{2z_{12}^4}+\frac{2T_w}{z_{12}^2}+\frac{T_w'}{z_{12}}, \quad T_w(z_1)U_4(z_2) = \frac{4U_4}{z_{12}^2}+\frac{U_4'}{z_{12}} \tag{2.5}$$

and both T_w and U_4 have regular OPEs with the $u(1)$ and $sl(N)$ currents $U(z)$ and $J_a^b(z)$

$$U(z_1)T_w(z_2) = U(z_1)U_4(z_2) = J_a^b(z_1)T_w(z_2) = J_a^b(z_1)U_4(z_2) = \text{regular}. \tag{2.6}$$

Thus, it follows from the above considerations that the currents $T_w(z), W(z)$ and $U_4(z)$ belong to the coset $W((sl(N+3),sl(3))/(u(1)\oplus sl(N))$.

3 W_3 contractions of $W(sl(N+3),sl(3))$ algebras

In this Section we would like to briefly recall the method for constructing the so called realizations modulo null fields for the W_3 algebra [6, 8, 9, 10, 11, 12, 13]. The distinguished feature of these realizations, with respect to the ordinary ones, consists in allowing the presence (besides standard terms) of a nonvanishing spin-4 *null* operator $\mathcal{V}$ in the OPE of the spin-3 current $\mathcal{W}$ with itself [3]

$$\mathcal{W}(z_1)\mathcal{W}(z_2) = \textit{standard terms} + \frac{\mathcal{V}}{z_{12}^2}+\frac{\mathcal{V}'}{2z_{12}} \quad . \tag{3.1}$$

The *null* operator $\mathcal{V}$ must satisfy the following requirement:

$$<\mathcal{V}\mathcal{V}>=0 \quad . \tag{3.2}$$

This corresponds to requiring that the OPE of the operator $\mathcal{V}$ with itself contains no central term. Obviously, being a null operator, $\mathcal{V}$ can generate only null fields in its OPEs. All such

[2] Due to the regular OPE of the spin-3 current $W(z)$ with $U(z)$ and $J_a^b(z)$, $W(z)$ is a primary current, not only with respect to T, but also with respect to T_w.

[3] The spin-4 operator $\mathcal{V}$ could be composite or elementary.

null operators span an ideal and can be consistently set to zero, yielding a W_3 algebra which closes only modulo null currents. Thus, on the shell defined by these constraints, the spin-2 and spin-3 currents form a W_3 algebra.

From the above definition it is clear that the problem of the construction of realizations modulo null fields is a very complicated one. However, it can be reduced to the easier task of constructing ordinary realizations, but for a larger $\mathcal{W}$-algebra, containing more currents than W_3 and including the OPE (3.1) within the full set of its OPEs. Then, for some discrete values of the central charge, the null field condition (3.2) for the spin-4 operator of such $\mathcal{W}$-algebra could be satisfied. Hence, for these specific values of the central charge, the realizations of such larger $\mathcal{W}$-algebra form simultaneously a realization of W_3 modulo null fields.

The $W(sl(N+3), sl(3))$ algebras constructed in the previous Section provides a class of such kind of $\mathcal{W}$-algebras, larger than W_3 and including in their OPEs a spin-4 current $\mathcal{V}$, according to (3.1). Having found the candidate algebras, all we need, in order to use the abovementioned approach for the construction of realizations of W_3 modulo null fields, is to determine the truncation condition of these class of $\mathcal{W}$-algebras to W_3, i.e. find the spectrum c_w corresponding to solutions of (3.2). As a second step, we must build $\mathcal{W}$-algebras realizations for the specific c_w values, for which $\mathcal{W}$ reduces to W_3. We start with nonlinear $W(sl(N+3), sl(3))$ algebras because they possess the following properties:

- the algebras $W(sl(N+3), sl(3))$ include the OPE (3.1);
- the algebras $W(sl(N+3), sl(3))$ can be linearized (see Section 4 below).

These facts are of use, because they respectively allow the following:

- to truncate (contract) the algebras $W(sl(N+3), sl(3))$ to W_3;
- to build realizations, starting from the linearizing $W(sl(N+3), sl(3))$ algebras.

Indeed, inspecting (2.1) it is easy to realize that the OPE of the spin-3 current with itself looks like (3.1). We stress that it is not possible to perform a redefinition of the currents of $W(sl(N+3), sl(3))$, such as to avoid the appearance of the $U_4(z)$ current in the r.h.s. of the OPE $W(z_1)W(z_2)$. Therefore, the $W(sl(N+3), sl(3))$ algebras do not contain W_3 as a subalgebra.

The vacuum expectation values for the currents T_w, W, U_4 which are contained in the coset $W((sl(N+3), sl(3))/(u(1) \oplus sl(N))$ read

$$
\begin{aligned}
< T_w T_w > &= -\frac{(2K+N)(-1+3K+2N)(1+4K+3N)}{2(K+N)(K+N+1)} \quad , \\
< WW > &= -\frac{(-2+3K+2N)(2+5K+4N)}{27K(3K+2N)} < T_w T_w > \quad , \\
< U_4 U_4 > &= \frac{8N(-1+2K+N)(-3+3K+2N)(1+3K+2N)(4K+3N)(3+6K+5N)}{120K^3-32K-32K^2-27N-59KN+230K^2N-27N^2+145KN^2+30N^3} \\
&\quad \times \frac{1}{3K^2(3K+2N)} < WW > \quad . \qquad (3.3)
\end{aligned}
$$

The spin-4 operator $U_4(z)$ becomes a null field - under the condition $< U_4 U_4 >= 0$, but requiring at the same time that $W(z)$ and $T_w(z)$ themselves do not turn into null fields, i.e. $< WW > \neq 0$

and $< T_w T_w > \neq 0$ - at the following values of the level parameter K and for the corresponding values of the central charge c_w :

$$K = \frac{1-N}{2}, (N>1) \quad ; \quad K = \frac{-1-2N}{3} \quad , \quad K = \frac{-3N}{4} \quad \Rightarrow \quad c_w = -2 \quad , \tag{3.4}$$

$$K = \frac{3-2N}{3} \quad \Rightarrow \quad c_w = \frac{2(N-6)(N+15)}{(N+3)(N+6)} \quad , \tag{3.5}$$

$$K = \frac{-3-5N}{6}, (N \neq 3) \quad \Rightarrow \quad c_w = \frac{2(N+5)(2N+3)}{N-3} \quad . \tag{3.6}$$

Hence, just for these values of the level parameter K, every realization of the algebras (2.1) induces a corresponding realization - modulo null fields - of the W_3 algebra formed by the currents T_w and W. All other poles and zeros of the vacuum expectation value $< U_4 U_4 >$ provide us with further contractions of the algebra, where the spin-3 current $W(z)$ and even the stress tensor T_w become null operators.

A list of the central charges of the spectrum (3.5) corresponding to some first values of N reads as follows:

$$c_w = -\frac{40}{7}, -\frac{17}{5}, -2, -\frac{38}{35}, -\frac{5}{11}, 0, \frac{22}{65}, \frac{46}{77}, \frac{4}{5}, \frac{25}{36} \quad . \tag{3.7}$$

Analogously, we can list the central charges of the spectrum (3.6) corresponding to first values of N as follows:

$$c_w = -30, -98, 198, 130, 110, 102, \frac{494}{5}, 98, \frac{690}{7} \quad . \tag{3.8}$$

A first attempt to classify the possible algebras which allow a contraction to W_N is made in [13] (see also references therein) where the central charge spectrum (3.5) was conjectured. However, the spectrum of central charges for the contraction of the $W(sl(N+3), sl(3))$ algebras to W_3 proposed in [13] is not exhaustive. In the present work the central charge spectrum (3.6) and the spectrum of K-values corresponding to the point $c_w = -2$ of (3.4) are constructed for the first time. A remark is in order concerning the latter point. Indeed this point of the resulting spectrum is particularly interesting, since it corresponds to three distinct representations of W_3 with arbitrary values of N. Namely, it turns out according to our result (3.4) that there exist three different null-fields realizations of W_3, which have the same value of the central charge $c_w = -2$.

The natural continuation and completion of our reasoning brings up the task of constructing explicitly the realizations of the $W(sl(N+3), sl(3))$ algebras, which in turn yield realizations of W_3 modulo null fields. The tool of the conformal linearization method of Refs. [2, 3, 4, 5] provides a direct way to construct such null-fields realizations, a fact that has been exploited for the first time in Ref. [6].

4 The linearizing $W(sl(N+3), sl(3))$ algebras

The method of the conformal linearization [2, 3, 4, 5] is a tool that drastically simplifies the problem of constructing explicitly the realizations - modulo null fields - of the W_3 algebra obtained by contracting the $W(sl(N+3), sl(3))$ algebras, according to the central charge

spectrum determined in the previous section. This is so, because any realization of the linearizing $W(sl(N+3),sl(3))$ algebras gives rise to a realization of the corresponding nonlinear $W(sl(N+3),sl(3))$ algebras.

The linearizing algebras for $W(sl(N+3),sl(3))$ can be constructed starting with the currents $(\mathcal{T},\mathcal{U},\mathcal{J}_a^b,\mathcal{U}_1,\mathcal{G}_a,\ \overline{\mathcal{G}}^a,\overline{\mathcal{Q}}^a,\mathcal{W})$. The OPEs of the conformally linearized $W(sl(N+3),sl(3))$ algebras read

$$
\begin{aligned}
\mathcal{T}(z_1)\mathcal{T}(z_2) &= \frac{c_l}{2z_{12}^4}+\frac{2\mathcal{T}}{z_{12}^2}+\frac{\mathcal{T}'}{z_{12}} \quad , \quad \mathcal{T}(z_1)\mathcal{J}_a^b(z_2)=\frac{\mathcal{J}_a^b}{z_{12}^2}+\frac{\mathcal{J}_a^{b\prime}}{z_{12}} \quad , \\
\mathcal{T}(z_1)\mathcal{G}_a(z_2) &= \frac{\mathcal{G}_a}{z_{12}^2}+\frac{\mathcal{G}_a{}'}{z_{12}} \quad , \quad \mathcal{T}(z_1)\overline{\mathcal{G}}^a(z_2)=\frac{\overline{\mathcal{G}}^a}{z_{12}^2}+\frac{\overline{\mathcal{G}}^{a}{}'}{z_{12}} \quad , \\
\mathcal{T}(z_1)\overline{\mathcal{Q}}^a(z_2) &= \frac{3(K+1+N)+3+N}{2(K+1+N)}\frac{\overline{\mathcal{Q}}^a}{z_{12}^2}+\frac{\overline{\mathcal{Q}}^{a}{}'}{z_{12}} \quad , \quad \mathcal{T}(z_1)\mathcal{U}(z_2)=\frac{\mathcal{U}}{z_{12}^2}+\frac{\mathcal{U}'}{z_{12}} \quad , \\
\mathcal{T}(z_1)\mathcal{W}(z_2) &= \frac{3(K+1+N)+3+N}{2(K+1+N)}\frac{\mathcal{W}}{z_{12}^2}+\frac{\mathcal{W}'}{z_{12}} \quad , \quad \mathcal{T}(z_1)\mathcal{U}_1(z_2)=\frac{\mathcal{U}_1}{z_{12}^2}+\frac{\mathcal{U}_1{}'}{z_{12}} \quad , \\
\mathcal{U}(z_1)\mathcal{U}(z_2) &= \frac{2(N+1)(K+1+N)}{(3+N)z_{12}^2} \quad , \quad \mathcal{U}(z_1)\overline{\mathcal{Q}}^a(z_2)=-\frac{\overline{\mathcal{Q}}^a}{z_{12}} \quad , \\
\mathcal{U}(z_1)\mathcal{W}(z_2) &= -\frac{\mathcal{W}}{z_{12}} \quad , \quad \mathcal{J}_a^b(z_1)\overline{\mathcal{Q}}^c(z_2)=-(\delta_a^c\overline{\mathcal{Q}}^b+\frac{1}{N}\delta_a^b\overline{\mathcal{Q}}^c)\frac{1}{z_{12}} \quad , \\
\mathcal{J}_a^b(z_1)\mathcal{J}_c^d(z_2) &= K(\delta_a^d\delta_c^b-\frac{1}{N}\delta_a^b\delta_c^d)\frac{1}{z_{12}^2}+(\delta_c^b\mathcal{J}_a^d-\delta_a^d\mathcal{J}_c^b)\frac{1}{z_{12}} \quad , \\
\mathcal{J}_a^b(z_1)\mathcal{G}_c(z_2) &= (\delta_c^b\mathcal{G}_a-\frac{1}{N}\delta_a^b\mathcal{G}_c)\frac{1}{z_{12}} \quad , \quad \mathcal{J}_a^b(z_1)\overline{\mathcal{G}}^c(z_2)=-(\delta_a^c\overline{\mathcal{G}}^b+\frac{1}{N}\delta_a^b\overline{\mathcal{G}}^c)\frac{1}{z_{12}} \quad , \\
\mathcal{U}_1(z_1)\mathcal{U}_1(z_2) &= \frac{KN}{(1+N)z_{12}^2} \quad , \quad \mathcal{U}_1(z_1)\overline{\mathcal{Q}}^a(z_2)=\frac{\overline{\mathcal{Q}}^a}{(N+1)z_{12}} \quad , \\
\mathcal{U}_1(z_1)\mathcal{G}_a(z_2) &= -\frac{\mathcal{G}_a}{z_{12}} \quad , \quad \mathcal{U}_1(z_1)\overline{\mathcal{G}}^a(z_2)=\frac{\overline{\mathcal{G}}^a}{z_{12}} \quad , \\
\mathcal{U}_1(z_1)\mathcal{W}(z_2) &= -\frac{N\mathcal{W}}{(N+1)z_{12}} \quad , \quad \mathcal{G}_a(z_1)\overline{\mathcal{Q}}^b(z_2)=-\frac{\delta_a^b\mathcal{W}}{z_{12}} \quad , \\
\mathcal{G}_a(z_1)\overline{\mathcal{G}}^b(z_2) &= \frac{K\delta_a^b}{z_{12}^2}+\left[\mathcal{J}_a^b-\frac{N+1}{N}\delta_a^b\mathcal{U}_1\right]\frac{1}{z_{12}} \quad , \quad \overline{\mathcal{G}}^b(z_1)\mathcal{W}(z_2)=-\frac{\overline{\mathcal{Q}}^a}{z_{12}} \quad .
\end{aligned}
\tag{4.1}
$$

In this basis the central charge of the linearizing algebras has the following expression:

$$
c_l=\frac{-6(K+1+N)^2+(N^2+2N+14)(K+1+N)+N^3+3N^2+2N+6}{K+1+N} \quad . \tag{4.2}
$$

The transformations that give the currents of the nonlinear $W(sl(N+3),sl(3))$ algebras in terms of the currents of the linearizing algebras (4.1) read[4]

$$
T = \mathcal{T}+\frac{(N+3)K\mathcal{U}'}{2(N+1)(K+1+N)}+\mathcal{U}_1{}' \quad ,
$$

[4]The expression for the W current can be easily obtained from the first pole of the OPE $G_a(z_1)\overline{G}^b(z_2)$ (2.1). It is a rather complicated and lengthy expression and we do not write it here explicitly.

$$
\begin{aligned}
U &= -\frac{2N\mathcal{U}}{N+1}+2\mathcal{U}_1 \quad , \quad J_a^b=\mathcal{J}_a^b+\frac{1}{N}\mathcal{U}_1\delta_a^b \quad , \quad G_a=\mathcal{G}_a \quad , \\
\overline{G}^a &= \overline{\mathcal{Q}}^a+h_1(\mathcal{T}\,\overline{\mathcal{G}}^a)+h_2(\mathcal{G}_b\,\overline{\mathcal{G}}^a\,\overline{\mathcal{G}}^b)+h_3(\mathcal{U}_1\,\mathcal{U}_1\,\overline{\mathcal{G}}^a)+h_4(\mathcal{U}_1{}'\,\overline{\mathcal{G}}^a)+h_5(\mathcal{U}_1\,\overline{\mathcal{G}}^{a\prime}) \\
&\quad +h_6(\mathcal{U}\,\mathcal{U}_1\,\overline{\mathcal{G}}^a)+h_7(\mathcal{U}\,\mathcal{U}\,\overline{\mathcal{G}}^a)+h_8(\mathcal{U}\,\overline{\mathcal{G}}^{a\prime})+h_9(\mathcal{U}'\,\overline{\mathcal{G}}^a)+h_{10}\overline{\mathcal{G}}^{a\prime\prime}+ \\
&\quad +h_{11}(\mathcal{J}_b^c\,\mathcal{J}_c^b\,\overline{\mathcal{G}}^a)+h_{12}(\mathcal{J}_b^c\,\mathcal{J}_c^a\,\overline{\mathcal{G}}^b)+h_{13}(\mathcal{U}\,\mathcal{J}_b^a\,\overline{\mathcal{G}}^b) \\
&\quad +h_{14}(\mathcal{J}_b^a\,\mathcal{U}_1\,\overline{\mathcal{G}}^b)+h_{15}(\mathcal{J}_b^{a\prime}\,\overline{\mathcal{G}}^b)+h_{16}(\mathcal{J}_b^a\,\overline{\mathcal{G}}^{b\prime}) \quad . \qquad (4.3)
\end{aligned}
$$

For the expressions of the coefficients h_1,...,h_{16} see Table 2 given in appendix.

With the above expressions we have completed our task to construct the realizations for the $W(sl(N+3), sl(3))$ (2.1) algebras, in terms of the currents of the linear algebras (4.1). In fact, starting from any given realization of (4.1), the use of (4.3) allows us to construct the realizations of the nonlinear algebras (2.1). Notice the appearence of the following exceptional value of the parameter K: $K=0$, which is a singular point of the transformations (4.3). Precisely for this value, the currents generating the starting nonlinear algebras (2.1) have to be redefined, in order to prevent the coefficients appearing in their OPEs from being singular, as it appears from some of the expressions given in Table 1.

In addition, there is a second level $K=-1-N$ where the transformations (4.3) have a singularity. However, for this point the nonlinear algebras we started with, i.e. (2.1), are not defined. Hence we must exclude this point from our present consideration of the linearizing $W(sl(N+3), sl(3))$ algebras as well. Thus, the set of exceptional points where the realizations of the $W(sl(N+3), sl(3))$ algebras develop some singular terms is entirely determined by the structure relations of the $W(sl(N+3), sl(3))$ algebras. Null field realizations of the W_3 algebra that lies in the coset $W((sl(N+3), sl(3))/(u(1)\oplus sl(N))$ are obtained from (4.3) at each value of the parameter K and the corresponding central charge c_w, given in eqs. (3.4)-(3.6).

5 Discussion and outlook

In this paper we have explicitly constructed the algebras $W(sl(N+3), sl(3))$. Using the relations between nonlinear and linearizing algebras, we have found their realizations, including the induced realizations of W_3 modulo null fields, when the algebras $W(sl(N+3), sl(3))$ are contracted to the W_3 one. Such null field realizations exist for the following values of the central charge of the W_3 algebra that lies in the coset $W((sl(N+3), sl(3))/(u(1)\oplus sl(N))$:

$$
c_w=-2, \quad \frac{2(N-6)(N+15)}{(N+3)(N+6)}, \quad \frac{2(N+5)(2N+3)}{N-3}. \qquad (5.1)
$$

It is interesting to observe that one can rewrite this spectrum in terms of the spectrum of W_3 minimal models, as follows:

$$
c_w=c_{W_3}^{min.mod.}(3,2), \quad c_{W_3}^{min.mod.}(5+N,2+N), \quad c_{W_3}^{min.mod.}(4-N,6), \qquad (5.2)
$$

where

$$
c_{W_3}^{min.mod.}(p,q) = 2\left[1-12\frac{(p-q)^2}{pq}\right] \quad . \qquad (5.3)
$$

We wish to conclude this Letter by predicting the spectrum of central charges for the $W(sl(N+3), sl(3))$ minimal models. One can start with noticing that the conformally linearized $W(sl(N+3), sl(3))$ algebras are homogeneous in the currents $\overline{\mathcal{Q}}^a, \mathcal{W}$. This remark implies that the latter are null fields and can be consistently set to zero $\overline{\mathcal{Q}}^a = \mathcal{W} = 0$. Inserting the above conditions into the expressions (4.3) leaves us with the Miura realization of the nonlinear $W(sl(N+3), sl(3))$ algebras in terms of the currents $\mathcal{T}$, $\mathcal{U}$, $\mathcal{J}_a^b$, $\mathcal{U}_1$, $\mathcal{G}_a$, $\overline{\mathcal{G}}^a$. It is possible to introduce a decoupling basis in the linearizing $W(sl(N+3), sl(3))$ algebras, with some redefined energy-momentum tensor T_{Vir}, which commutes with the remaining currents and possesses the following central charge:

$$c_{Vir} = 1 - 6\frac{(K-1)^2}{K} \quad . \tag{5.4}$$

The values of c_{Vir} corresponding to the minimal models of the Virasoro algebra at

$$K = \frac{p}{q} \Rightarrow c_{Vir} = 1 - 6\frac{(p-q)^2}{pq}$$

induce the following spectrum for the central charge (2.2) of the nonlinear $W(sl(N+3), sl(3))$ algebras:

$$c^{min.mod.}_{W(sl(N+3),sl(3))} = N^2 + 26N + 50 - \frac{24p^2 + (N+4)(N+3)(N+2)q^2}{pq} \quad . \tag{5.5}$$

In all known cases of linearizing algebras the minimal models the Virasoro algebra spanned by T_{Vir} reproduce the minimal models for the corresponding nonlinear algebras. For instance, in the case of $N = 2$ superconformal and $W_3^{(2)}$ algebras, it can be checked that the above procedure yields the spectrum of minimal models [5]. The values of c_{Vir} corresponding to the minimal models of the Virasoro algebra (5.4) are also found to induce the central charges $c^{min.mod.}_{W_N}$ for the W_N algebras [5]. Using this argument, it seems reasonable to expect that this feature persists for $W(sl(N+3), sl(3))$ algebras, as well. Nonetheless, it is needed that our conjecture in eq. (5.5) for the spectrum of central charges for the minimal models of the $W(sl(N+3), sl(3))$ algebras be checked by standard methods.

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Appendix

Next, we write down the expressions of the coefficients in the OPEs (2.1) for the algebra $W(sl(N+3), sl(3))$.

$c_t = \frac{1}{R_1}(2R_1-24K^2-22KN-6N^2+KN^2)$	$c_u = \frac{4NR_2}{3+N}$	$c_g = -\frac{(2K+N)R_2}{3R_1}$	$\epsilon = \frac{(2-R_2)(2R_1+R_2)}{9KR_2}$
$\alpha_1 = \frac{(3+N)(2K+N)}{6NR_1}$	$\beta_1 = \frac{(R_2-2)(R_2-1)}{18K(K+N)R_2^2} \times (R_1+R_2)(2R_1+R_2)$	$\epsilon_1 = \frac{2N}{3K\epsilon}$	$\epsilon_{14} = \frac{(3+N)(-3+K-N)}{144KNR_1\epsilon}$
$\alpha_2 = -\frac{(2K+N)R_2}{3KR_1}$	$\beta_2 = \frac{(R_2-2)(2R_1+R_2)(KR_2-R_1)}{18K(K+N)R_2^2}$	$\epsilon_2 = -\frac{2(3+N)}{3KR_2\epsilon}$	$\epsilon_{15} = \frac{(3+N)}{108KR_2\epsilon} \times \frac{(6R_1+6K^2+7KN+2N^2)}{R_1}$
$\alpha_3 = -\frac{(3+N)(2+N)}{24N^2R_1}$	$\beta_3 = \frac{(2-R_2)(3+N)(2R_1+R_2)}{12KNR_2^2}$	$\epsilon_3 = -\frac{4}{3K\epsilon}$	$\epsilon_{16} = \frac{2(3+N)}{27K^2N(K+N)R_1\epsilon}$
$\alpha_4 = -\frac{R_2}{3KR_1}$	$\beta_4 = \frac{(R_2-2)(2R_1+R_2)}{6K(K+N)R_2}$	$\epsilon_4 = \frac{2R_1}{9KR_2\epsilon}$	$\epsilon_{17} = \frac{(3+N)(6K+2N+KN)}{18K^2NR_1R_2\epsilon}$
$\alpha_5 = \frac{2-K}{6KR_1}$	$\beta_5 = -\frac{2R_1}{3KR_2}$	$\epsilon_5 = \frac{(2-K)(3+N)}{18KNR_2^2\epsilon}$	$\epsilon_{18} = \frac{(3+N)}{27K^2N(K+N)} \times \frac{(R_2^2+3KR_2+3KNR_1)}{R_1R_2\epsilon}$
$\alpha_6 = -\frac{3+N}{6NR_2}$	$\beta_6 = \frac{(3+N)(2NR_1+R_2)}{12KN^2R_2^2}$	$\epsilon_6 = \frac{2(2-K)}{9K^2R_2\epsilon}$	$\epsilon_{19} = -\frac{1}{9K^2(K+N)R_1\epsilon}$
$\alpha_7 = \frac{(3+N)^2}{144N^3R_1R_2^2} \times (2NR_1+NR_2+2R_2)$	$\beta_7 = -\frac{3+N}{3KNR_2}$	$\epsilon_7 = -\frac{3+N}{9KR_2\epsilon}$	$\epsilon_{20} = \frac{2-4R_1+K^2+KN}{18K^2(K+N)R_1R_2\epsilon}$
$\alpha_8 = \frac{(3+N)}{36NR_1R_2} \times (3R_1+6K^2+7KN+2N^2)$	$\beta_8 = -\frac{3+N}{12KN}$	$\epsilon_8 = -\frac{3+N}{9KR_2\epsilon}$	$\epsilon_{21} = -\frac{2K+N}{3K^2(K+N)R_1\epsilon}$
$\alpha_9 = \frac{(K-2)(3+N)}{12KNR_1R_2}$	$\beta_9 = \frac{(2-K)(3+N)(2+R_2)}{12KNR_2{}^2}$	$\epsilon_9 = -\frac{1}{12\epsilon}$	$\epsilon_{22} = \frac{3K^2R_1-12K^2+4N^2}{36K^2(K+N)R_1\epsilon}$
$\alpha_{10} = \frac{1}{9KR_1(K+N)}$	$\beta_{10} = \frac{1}{3K(K+N)}$	$\epsilon_{10} = \frac{(3+N)^2}{864KN^3R_1R_2{}^3\epsilon}(3K^2N-18K-24NR_1-2KN^2-4N^3)$	$\epsilon_{23} = \frac{1}{36K^2(K+N)R_1R_2\epsilon} \times (-20K-16K^2-35K^3+9K^4-16N-32KN-78K^2N+15K^3N-16N^2-54KN^2+6K^2N^2-12N^3)$
$\alpha_{11} = \frac{R_2-3K^2-3KN}{18KR_1(K+N)}$	$\beta_{11} = \frac{R_1}{3K(K+N)R_2}$	$\epsilon_{11} = \frac{(3+N)^2}{72KN^2R_1R_2^2\epsilon} \times (6K+6N+5KN+4N^2)$	
$\alpha_{12} = \frac{3+N}{6KNR_1}$	$\beta_{12} = \frac{R_2}{6K(K+N)}$	$\epsilon_{12} = \frac{(3+N)}{36K^2N^2R_1R_2^2\epsilon}(K^2N-18K-8N-10KN-4N^2)$	
$\alpha_{13} = -\frac{1}{6KR_1}$	$\beta_{13} = \frac{(K-2)(2+R_2)}{6K(K+N)R_2}$	$\epsilon_{13} = \frac{3+N}{432KNR_1R_2^2\epsilon}(27K^3-60-48K-105K^2-72N-150KN+3K^2N-60N^2-38KN^2-16N^3)$	
$\alpha_{14} = -\frac{2K+N}{3KR_1}$	$\beta_{14} = \frac{1}{36K(K+N)R_2{}^2}(4+8K+13K^2+9K^3+18K^4+8N+38KN+39K^2N+33K^3N+20N^2+46KN^2+20N^2K^2+16N^3+4KN^3)$		
$\alpha_{15} = -\frac{R_1+2K^2+KN}{6KR_1}$			

Table 1

Here we define $R_1 = 1+N+K$ and $R_2 = 3K+2N$.

Finally, we give the coefficients for the currents in the nonlinear basis (4.3).

$h_1 = \frac{1}{3K}$	$h_2 = -\frac{1}{3KR_1}$	$h_3 = -\frac{2+N+N^2}{6KN^2R_1}$	$h_4 = \frac{-2+2K-N-N^2}{6KNR_1}$
$h_5 = \frac{1+2K+3N}{3KNR_1}$	$h_6 = \frac{3+N}{3KN(1+N)R_1}$	$h_7 = -\frac{(2+N)(3+N)}{6K(1+N)^2R_1}$	$h_8 = -\frac{(3+N)(K+N)}{3K(1+N)R_1}$
$h_9 = -\frac{(3+N)(K+N)}{6K(1+N)R_1}$	$h_{10} = -\frac{R_1+2(K+N)^2}{6KR_1}$	$h_{11} = -\frac{1}{6KR_1}$	$h_{12} = -\frac{1}{3KR_1}$
$h_{13} = -\frac{3+N}{3K(1+N)R_1}$	$h_{14} = \frac{2}{3KNR_1}$	$h_{15} = \frac{1-K}{3KR_1}$	$h_{16} = \frac{1-2R_1}{3KR_1}$

Table 2

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